

# AN UPPER BOUND ON STICK NUMBERS OF KNOTS

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ABSTRACT. In 1991, Negami found an upper bound on the stick number  $s(K)$  of a nontrivial knot  $K$  in terms of the minimal crossing number  $c(K)$  of the knot which is  $s(K) \leq 2c(K)$ . In this paper we improve this upper bound to  $s(K) \leq \frac{3}{2}(c(K) + 1)$ . Moreover if  $K$  is a non-alternating prime knot, then  $s(K) \leq \frac{3}{2}c(K)$ .

## 1. INTRODUCTION

A simple closed curve embedded into the Euclidean 3-space is called a *knot*. Two knots  $K$  and  $K'$  are said to be *equivalent*, if there exists an orientation preserving homeomorphism of  $\mathbb{R}^3$  which maps  $K$  to  $K'$ , or to say roughly, we can obtain  $K'$  from  $K$  by a sequence of moves without intersecting any strand of the knot. And the equivalence class of  $K$  is called the *knot type* of  $K$ . A knot equivalent to another knot in a plane of the 3-space is said to be *trivial*. A *stick knot* is a knot which consists of finite line segments, called *sticks*.

One natural question concerning stick knots may be the *stick number*  $s(K)$  of a knot  $K$  which is defined to be the minimal number of sticks for construction of the knot type into a stick knot. Since this representation of knots has been considered to be a useful mathematical model of cyclic molecules or molecular chains, the stick number may be an interesting quantity not only in knot theory of mathematics, but also in chemistry and physics. Although it seems to be not easy to determine  $s(K)$  completely for arbitrary knot  $K$ , which is usual for any other minimality invariants of knots, there are some literatures in which the range of  $s(K)$  was theoretically investigated [2, 10, 12, 15, 16]. Especially, in 1991, Negami found upper and lower bounds on the stick number of any nontrivial knot  $K$  in terms of the crossing number  $c(K)$  [15]:

$$\frac{5 + \sqrt{25 + 8(c(K) - 2)}}{2} \leq s(K) \leq 2c(K)$$

Here the crossing number  $c(K)$  is the minimal number of double points in any generic projection of the knot type into the plane  $\mathbb{R}^2 \subset \mathbb{R}^3$ . In [1, 10] it was questioned whether it is possible to improve the Negami's inequalities: To describe specifically,

- Q1. Is there any knot satisfying  $2s(K) = 5 + \sqrt{25 + 8(c(K) - 2)}$  ?
- Q2. Is there any knot satisfying  $s(K) = 2c(K)$  other than the trefoil knots?

The Negami's lower bound was slightly improved by Calvo [5]. And recently Elifai showed that the answer for the first question is negative for the knots with  $c(K) \leq 26$  [9].

In this paper we give an improved upper bound on  $s(K)$  and answer for the second question. The following theorem is the main result of this paper.

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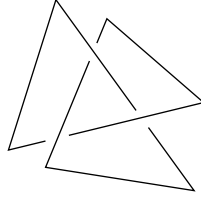


FIGURE 1. A projection, a diagram and a stick representation of the right-handed trefoil

**Theorem 1.** *Let  $K$  be any nontrivial knot. Then  $s(K) \leq \frac{3}{2}(c(K) + 1)$ . Moreover if  $K$  is a non-alternating prime knot, then  $s(K) \leq \frac{3}{2}c(K)$ .*

Note that  $c(K)$  is at least three for any nontrivial knot  $K$ . If  $\frac{3}{2}(c(K) + 1) \geq 2c(K)$ , then  $c(K) \leq 3$ . It is known that the *right-handed trefoil knot* in Figure 1 and its mirror image (called *left-handed trefoil*) are the only knot types with  $c(K) = 3$ . And their stick numbers are exactly six. Therefore, from our theorem, we can conclude that the second question is negative for any knot type other than the trefoil knots.

In the rest of this paper we will prove Theorem 1 by investigating the relation among stick number  $s(K)$ , crossing number  $c(K)$  and another minimality quantity, *arc index*  $a(K)$ .

## 2. PROOF OF MAIN THEOREM

We introduce some definitions necessary for the proof of Theorem 1. A continuous map from the unit circle into  $\mathbb{R}^2$  is called a *knot projection*, if each multiple point of the map is a transversal double point which will be called a *crossing*. By adding the information on the relative height of two strands at each crossing into the projection, we obtain a *diagram* representing a knot. An example of a diagram is given in Figure 1. The *crossing number*  $c(K)$  of a knot  $K$  is defined to be the minimal number of crossings among all diagrams representing the knot type.

In our proof we consider a specific type of knot diagram which is obtained by drawing  $n$  chords  $l_1, \dots, l_n$  on a 2-dimensional disk  $B$  according to the following rules:

- (1) The end points of each  $l_i$  lie on the boundary of  $B$ .
- (2) If  $l_i$  and  $l_j$  share a crossing in the interior of  $B$  and  $i < j$ , then  $l_i$  underpasses  $l_j$ .

If a diagram of such type represents a knot  $K$ , it is called an *arc presentation* of  $K$ . And the *arc index*  $a(K)$  of a knot  $K$  is defined to be the minimal number of chords among all possible arc presentations of its knot type. In fact our definition of arc presentation is a little modified from the original one, but essentially identical [4, 6]. The left of Figure 2 shows an arc presentation of the trefoil knot.

Bae and Park established an upper bound of arc index in terms of the crossing number. Corollary 4 and Theorem 9 in [3] provide the following;

**Theorem 2** (Bae and Park). *Let  $K$  be any nontrivial knot. Then  $a(K) \leq c(K) + 2$ . Moreover if  $K$  is a non-alternating prime knot, then  $a(K) \leq c(K) + 1$ .*

Therefore, if we prove Theorem 3, then the proof of our main theorem is completed.

**Theorem 3.** *Let  $K$  be any nontrivial knot. Then  $s(K) \leq \frac{3}{2}(a(K) - 1)$ .*

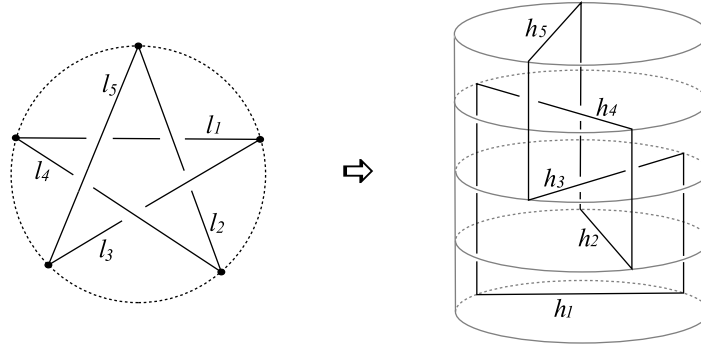


FIGURE 2. A stick knot in cylinder constructed from an arc presentation

*Proof.* Let  $K$  be a nontrivial knot with  $a(K) = n$  and  $D$  be an its arc presentation with  $n$  chords  $l_1, \dots, l_n$ .  $\overline{D}$  denotes the projection of  $D$ . Let  $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the map defined by  $\pi(x, y, z) = (x, y)$ . From  $D$  we construct a stick knot  $K_1$  in the cylinder  $B \times [1, n]$  so that  $\pi(K_1) = \overline{D}$ . For each integer  $i \in [1, n]$ , put a line segment  $h_i$  into  $B \times \{i\}$  so that  $\pi(h_i) = l_i$ . If  $l_i \cap l_j \cap \partial B = \{p\}$ , then connect  $\pi^{-1}(p) \cap h_i$  to  $\pi^{-1}(p) \cap h_j$  by a vertical line segment  $v_{ij}$  so that  $\pi(v_{ij}) = p$ . Note that we do not distinguish  $v_{ij}$  from  $v_{ji}$ . By adding all such vertical sticks, we obtain a stick knot  $K_1$  with  $2n$  sticks which is equivalent to  $K$ . Figure 2 shows an example of a stick knot constructed from an arc presentation of the right-handed trefoil.

A horizontal stick  $h_i$  is said to be *type-I* (resp. *type-III*), if the indices of the two chords adjacent to  $l_i$  in  $D$  are greater (resp. less) than  $i$ . If neither type-I nor type-III, then  $h_i$  is *type-II*. Notice that  $h_1$  and  $h_n$  should be type-I and type-III, respectively. If  $h_{n-1}$  is type-II, then we can modify  $K_1$  as illustrated in Figure 3(a), so that the number of horizontal sticks is reduced by one, which is contradictory to the minimality of the number of chords. Since  $K_1$  is a nontrivial knot,  $h_{n-1}$  can not be type-I. Hence  $h_{n-1}$  should be type-III and similarly  $h_2$  should be type-I.

From  $K_1$  we construct another stick knot  $K_2$  in which the  $z$ -coordinate of each  $h_i$  may be changed into some integer  $z_i$ , while its  $xy$ -coordinates are preserved. Concretely, if we denote the  $i$ -th horizontal stick of  $K_2$  by  $h'_i$ , then  $\pi(h_i) = \pi(h'_i)$  and  $h'_i \subset B \times \{z_i\}$  in  $B \times [0, \infty)$ . The height  $z_i$  will be determined in inductive way. Firstly, set  $z_1 = 1$  and  $z_2 = 2$ . For  $3 \leq i \leq n$ , if  $h_i$  is type-I, then  $z_i$  is set to be  $z_{i-1} + 1$ . If  $h_i$  is type-II, there is a vertical stick  $v_{ij}$  with  $j < i$  which is adjacent to  $h_i$ . Then put  $h'_i$  into  $B \times \{z_i\}$  for some large enough  $z_i$  and connect  $h'_i$  to  $h'_j$  via the vertical stick  $v'_{ij}$  between  $B \times \{z_i\}$  and  $B \times \{z_j\}$ , so that the interior of the triangle determined by  $h'_i \cup v'_{ij}$  has no intersection with any other horizontal stick  $h'_k$ ,  $k < i$ . Here, such a triangle will be called a *reducible triangle* of  $h'_i$ . If  $h_i$  is type-III, that is,  $h_i$  is adjacent to some  $v_{ij}$  and  $v_{ik}$  with  $i > j > k$ , then similarly the height of  $h'_i$  is determined so that the triangle whose boundary contains  $h'_i \cup v'_{ij}$  is reducible.

Now we modify  $K_2$  in purpose to decrease the number of sticks. For each  $i$  from 3 to  $n-1$ , if  $h'_i$  is type-II or III, replace  $h'_i \cup v'_{ij}$  with the other edge of the reducible triangle (See Figure 3(b)). Since the interior of the triangle has no intersection with any other part of the knot, such replacement preserves the knot type. And the number of sticks becomes reduced by one, after each modification. For  $h'_n$ , we modify the knot in another way. Let  $v'_{ni}$  and  $v'_{nj}$  be the sticks adjacent to  $h'_n$ . The other stick adjacent to  $v'_{ni}$  (resp.  $v'_{nj}$ ) is denoted by  $e_i$  (resp.  $e_j$ ). Extend  $e_i$  and  $e_j$  toward the end points  $e_i \cap v'_{ni}$  and  $e_j \cap v'_{nj}$ , respectively, long enough so

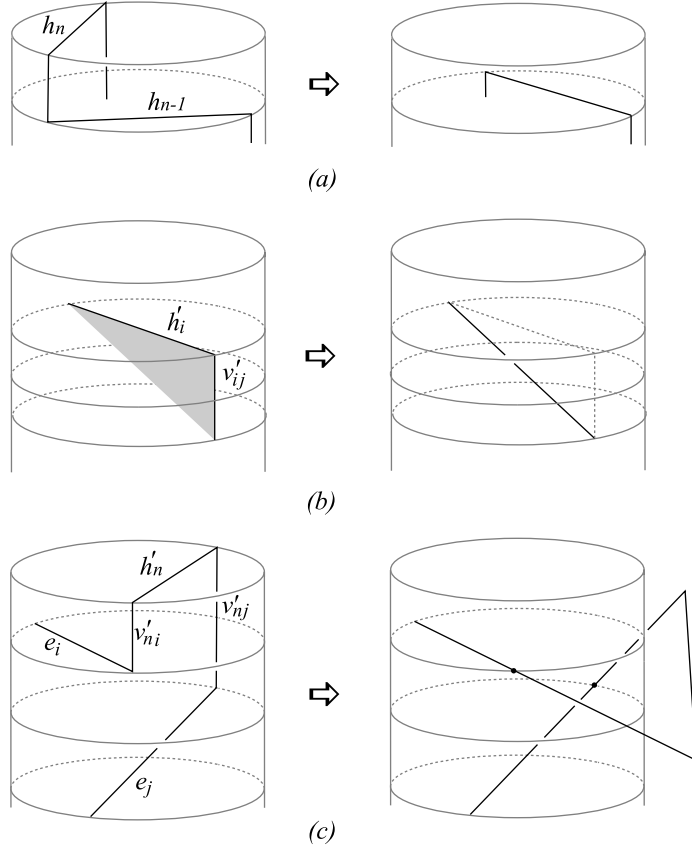


FIGURE 3. (a) Reduction when  $h_{n-1}$  is type-II, (b) Reduction along a reducible triangle and (c) Reduction near  $h'_n$

that the two extended line segments are connected by a line segment outside of  $B \times [1, z_n]$ . Replace  $e_i \cup v'_{ni} \cup h'_n \cup v'_{nj} \cup e_j$  with these three line segments (See Figure 3(c) for example). Then the knot type is preserved, but the number of sticks is reduced by two. Let  $K_3$  be the resulting stick knot.

Let  $\beta_1(K_1)$ ,  $\beta_2(K_1)$  and  $\beta_3(K_1)$  be the numbers of type-I, type-II and type-III horizontal sticks of  $K_1$ , respectively. Note that  $\beta_1(K_1) = \beta_3(K_1)$ . Since  $n = \beta_1(K_1) + \beta_2(K_1) + \beta_3(K_1)$ , the number of sticks of  $K_3$  is equal to

$$2n - \beta_2(K_1) - (\beta_3(K_1) - 1) - 2 = n + \beta_1(K_1) - 1 \quad .$$

Therefore,

$$s(K) \leq n + \beta_1(K_1) - 1 \quad .$$

Now we consider an upper bound of  $\beta_1(K_1)$ . If  $n$  is odd, then  $\beta_1(K_1) \leq (n-1)/2$ . If  $n$  is even, then  $\beta_1(K_1) \leq n/2$  in which the equality holds only when  $\beta_2(K_1) = 0$ . In that case, let  $v_{i1}$  and  $v_{j1}$  be the horizontal sticks adjacent to  $h_1$  in  $K_1$ . And replace  $v_{i1} \cup h_1 \cup v_{j1}$  with  $v_{i(n+1)} \cup h_{n+1} \cup v_{j(n+1)}$ , where  $h_{n+1}$  is the horizontal line segment in  $B \times \{n+1\}$  satisfying  $\pi(h_1) = \pi(h_{n+1})$ . Then the resulting stick knot  $K'_1$  in  $B \times [2, n+1]$  is equivalent to  $K_1$ .

Because  $\beta_2(K_1) = 0$ , we have

$$\beta_1(K'_1) = \beta_1(K_1) - 1 \leq \frac{n}{2} - 1 < \frac{n-1}{2} \quad .$$

To summarize, for a nontrivial knot  $K$  with  $a(K) = n$ , there exists an equivalent stick knot  $K'$  with  $2n$  sticks in the cylinder satisfying

$$\beta_1(K') \leq \frac{n-1}{2}$$

and therefore

$$s(K) \leq n + \beta_1(K') - 1 \leq a(K) + \frac{a(K) - 1}{2} - 1 = \frac{3}{2}(a(K) - 1) \quad .$$

□

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